

5.3

Definite Integrals and Antiderivatives

Properties of Definite Integrals

Table 5.3 Rules for Definite Integrals

1. *Order of Integration:* $\int_b^a f(x) dx = -\int_a^b f(x) dx$ A definition

2. *Zero:* $\int_a^a f(x) dx = 0$ Also a definition

3. *Constant Multiple:* $\int_a^b kf(x) dx = k \int_a^b f(x) dx$ Any number k

$$\int_a^b -f(x) dx = -\int_a^b f(x) dx \quad k = -1$$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

5. *Additivity:* $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

6. *Max-Min Inequality:* If $\max f$ and $\min f$ are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b - a) \leq \int_a^b f(x) dx \leq \max f \cdot (b - a).$$

7. *Domination:* $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x) dx \geq \int_a^b g(x) dx$

$$f(x) \geq 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x) dx \geq 0 \quad g = 0$$

EXAMPLE 1 Using the Rules for Definite Integrals

Suppose

$$\int_{-1}^1 f(x) dx = 5, \quad \int_{-1}^4 f(x) dx = -2, \quad \text{and} \quad \int_{-1}^1 h(x) dx = 7.$$

Find each of the following integrals, if possible.

(a) $\int_4^1 f(x) dx$

(b) $\int_{-1}^4 f(x) dx$

(c) $\int_{-1}^1 [2f(x) + 3h(x)] dx$

(d) $\int_0^1 f(x) dx$

(e) $\int_{-2}^2 h(x) dx$

(f) $\int_{-1}^4 [f(x) + h(x)] dx$

a) 2

b) $\int_{-1}^1 f(x) dx + \int_1^4 f(x) dx = 3$

c) $2 \int_{-1}^1 f(x) dx + 3 \int_{-1}^1 h(x) dx$
 $2(5) + 3(7) = 31$

d) cannot do

e) cannot do

f) cannot do

$$\min f(4-2) \leq \underbrace{\int_2^4 f(x) dx}_{7} \leq \max f(4-2)$$

$$7 \leq \max f(4-2)$$

$$3.5 \leq \max f$$

EXAMPLE 2 Finding Bounds for an Integral

Show that the value of $\int_0^1 \sqrt{1 + \cos x} \, dx$ is less than $3/2$.

Consider, then, what happens if we take a large *sample* of n numbers from regular subintervals of the interval $[a, b]$. One way would be to take some number c_k from each of the n subintervals of length

$$\Delta x = \frac{b-a}{n}.$$

The average of the n sampled values is

$$\begin{aligned} \frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} &= \frac{1}{n} \cdot \sum_{k=1}^n f(c_k) \\ &= \frac{\Delta x}{b-a} \sum_{k=1}^n f(c_k) \quad \frac{1}{n} = \frac{\Delta x}{b-a} \\ &= \frac{1}{b-a} \cdot \sum_{k=1}^n f(c_k) \Delta x. \end{aligned}$$

DEFINITION Average (Mean) Value

If f is integrable on $[a, b]$, its **average (mean) value** on $[a, b]$ is

$$av(f) = \frac{1}{b-a} \int_a^b f(x) dx.$$

EXAMPLE 3 Applying the Definition

Find the average value of $f(x) = 4 - x^2$ on $[0, 3]$. Does f actually take on this value at some point in the given interval?

$$\frac{1}{3-0} \int_0^3 f(x) dx$$

$$\frac{1}{3} (3) = 1$$

$$4 - x^2 = 1$$

$$-x^2 = -3$$

$$x^2 = 3$$

$$x = \pm \sqrt{3}$$

THEOREM 3 The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

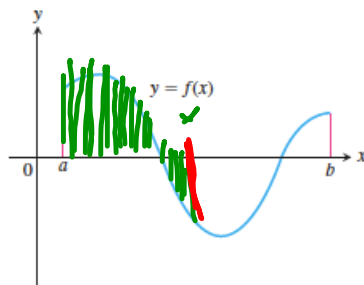


Figure 5.27 The graph of the function in Exploration 2.

EXPLORATION 2 Finding the Derivative of an Integral

Group Activity Suppose we are given the graph of a continuous function f , as in Figure 5.27.

1. Copy the graph of f onto your own paper. Choose any x greater than a in the interval $[a, b]$ and mark it on the x -axis.
2. Using only *vertical line segments*, shade in the region between the graph of f and the x -axis from a to x . (Some shading might be below the x -axis.)
3. Your shaded region represents a definite integral. Explain why this integral can be written as $\int_a^x f(t) dt$. (Why don't we write it as $\int_a^x f(x) dx$?)
4. Compare your picture with others produced by your group. Notice how your integral (a real number) depends on which x you chose in the interval $[a, b]$. The integral is therefore a *function of x* on $[a, b]$. Call it F .
5. Recall that $F'(x)$ is the limit of $\Delta F / \Delta x$ as Δx gets smaller and smaller. Represent ΔF in your picture by drawing *one more vertical shading segment* to the right of the last one you drew in step 2. ΔF is the (signed) *area* of your vertical segment.
6. Represent Δx in your picture by moving x to beneath your newly-drawn segment. That small change in Δx is the *thickness* of your vertical segment.
7. What is now the *height* of your vertical segment?
8. Can you see why Newton and Leibniz concluded that $F'(x) = f(x)$?

If all went well in Exploration 2, you concluded that the derivative with respect to x of the integral of f from a to x is simply f . Specifically,

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

This means that the integral is an *antiderivative* of f , a fact we can exploit in the following way.

If F is any antiderivative of f , then

$$\int_a^x f(t) dt = F(x) + C$$

for some constant C . Setting x in this equation equal to a gives

$$\int_a^a f(t) dt = F(a) + C$$

$$0 = F(a) + C$$

$$C = -F(a).$$

Putting it all together,

$$\int_a^x f(t) dt = F(x) - F(a).$$

EXAMPLE 4 Finding an Integral Using Antiderivatives

Find $\int_0^\pi \sin x \, dx$ using the formula $\int_a^x f(t) \, dt = F(x) - F(a)$.